celeration. The expression which he thus obtains for the first two terms of this acceleration, is

$$-\left(\frac{3}{2}m^2-\frac{3771}{64}m^4\right)\int(e'^2-E'^2)ndt.$$

According to Plana, the corresponding expression is

$$-\left(\frac{3}{2}m^2-\frac{2187}{128}m^4\right)\int(e'^2-E'^2)ndt.$$

It will be observed that the coefficient of the second term has been completely altered in consequence of the introduction of the new terms.

The numerical effect of this alteration is to diminish by $1''\cdot66$ the coefficient of the square of the time in the expression for the secular acceleration; the time being, as usual, expressed in centuries.

It will, of course, be necessary to carry the approximation much further in order to obtain such a value of this coefficient as may be employed with confidence in the calculation of ancient eclipses.

In conclusion, the author states, that the existence of the new terms in the expression of the moon's coordinates occurred to him some time since, when he was engaged in thinking over a new method of treating the lunar theory, though he did not then perceive their important bearing on the secular equation. His attention was first directed to this subject while endeavouring to supply an omission in the Theory of the Moon given by Pontécoulant in his 'Théorie Analytique.' In this valuable work, the author, following the example originally set by Sir J. Lubbock in his tracts on the lunar theory, obtains directly the expressions for the moon's coordinates in terms of the time, which are found in Plana's theory by means of the reversion of series. With respect to the secular acceleration of the mean motion, however, Pontécoulant unfortunately adopts Plana's result without examination. On performing the calculation requisite to complete this part of the theory, the author was surprised to find that the second term of the expression for the secular acceleration thus obtained, not only differed totally in magnitude from the corresponding term given by Plana, but was even of a contrary sign. His previous researches, however, immediately led him to suspect what was the origin of this discordance, and when both processes were corrected by taking into account the new terms whose existence he had already recognized, he had the satisfaction of finding a perfect agreement between the results.

3. "On a Theory of the conjugate relations of two rational integral functions, comprising an application to the Theory of Sturm's Functions, and that of the greatest Algebraical Common Measure." By J. J. Sylvester, Esq., M.A., F.R.S., Barrister at Law. Received June 16, 1853.

The memoir consists of four sections. In the first section, the theory of the residues obtained by applying the process of the common measure to two algebraical functions is discussed. It is shown that

a certain superfluous or *allotrious* factor enters into each, the value of which, in terms of the leading coefficients of the residues in their simplified form, is determined; and the simplified residues themselves are subsequently obtained from the given functions by a direct method.

In the case where the two functions are of the same degree (m) in x, m functions of the degree $m-1$ in x are formed, which, being identical with those employed in the process which goes by the name of Bezout's abridged method, the author terms the Bezoutics or Bezoutic primaries. By linear elimination performed between these, a second system of functions, whose degrees in x extend from $m-1$ to 0, are formed, which he terms the Bezoutic secondaries: these Bezoutic secondaries are proved to be identical with the simplified residues. A similar theory is shown to be applicable in the general case of the functions being of unlike degrees. Other modes of obtaining the simplified residues by a direct method are also given. The coefficients of the primary system of Bezoutics form a square symmetrical about one axis, to which (as to every symmetrical matrix) a certain homogeneous quadratic function of (m) variables is appurtenant. This quadratic function is termed the Bezoutiant, the properties of which are discussed in the fourth section.

Every residue is what may be termed a syzygetic function or conjunctive of the two given functions; these being respectively multiplied by certain appropriate rational integral functions, their sum may be made to represent a residue. These multipliers are termed the syzygetic multipliers; and they form two series, one corresponding to the successive numerators, the other to the successive denominators of the convergents to the algebraical continued fraction which expresses the ratio of the two given functions. The residues are obviously a particular class of the conjunctives that can be formed from the given functions; every conjunctive has the property of vanishing when the two functions to which it is appurtenant vanish simultaneously; and in general, for any given degree in x, an infinite number of such conjunctives can be formed.

In the second section, the author commences with obtaining in terms of the roots and factors of the two given functions, a variety of forms, all containing *arbitrary* forms of function in their several terms, and representing a conjunctive of any degree not exceeding the sum of the degrees of the two given functions in its most general form. The author then reverts to the Bezoutic system of the first section, and obtains the general solution for the conjunctive of any given degree in x in terms of the *coefficients* of the given function; by aid of this general solution he demonstrates that the residues obtained by the common measure process (divested of their allotrious factors), are the conjunctives of the lowest *weight* in the roots of the given functions for their several degrees; and obtains the value of this weight. He then demonstrates that certain rational but fractional forms ascribed to the arbitrary functions in the general expressions for a conjunctive in terms of the roots, will make these expressions integral and of the minimum weight; they will all be

consequently identical (save as to a numerical factor) with one another, and with the simplified residues. The formulæ thus obtained for the simplified residues deserve particular attention on their own account, being double sums of terms, any single series of which is made up of fractions whose denominators are the products of the differences between a certain number of the roots of each one of the functions and a certain other number of the same combined in every possible manner, thus containing a vast extension of the ordinary theory of partial fractions. The author subsequently determines under a similar form, the value of each of the multipliers which connects the given functions syzygetically with the simplified residues, and establishes a general theorem of reciprocity, by aid of certain general properties of continued fractions, between the series of residues and either series of syzygetic multipliers.

The third section is divided into two parts. The first part is devoted to a determination of the values of the preceding formulæ in the case to which Sturm's theorem refers, where one of the given functions is the first differential derivative of the other; when this is the case the roots and factors of the second function are functions of those of the first, and it will be found that one of the polymorphic representations for the residue of any given degree will consist of terms, each of which is convertible into an integral function of the roots and factors of the given primitive function; in this way are obtained the author's well-known formulæ for Sturm's auxiliary functions. In like manner, the multiplier which affects the derivative function in the syzygy between the primitive, the derivative and any simplified residue, may also be expressed immediately as a sum of integral functions of the roots and factors of the primitive, complementary in some sort to the formulæ for the residues. The formula for the remaining syzygetic multiplier, (that which attaches to the primitive itself,) cannot be obtained directly by a similar method, but it is deduced by aid of the syzygetic equation itself, all the other of the five terms of which are known, or have been previously determined. The process of obtaining this last-named multiplier is one of great peculiarity and interest, and results in a form far more complex than that for the residues or for the other syzygetic multiplier.

In the second part of the third section are contained some curious and valuable expressions for the residues and multipliers, communicated to the author by M. Hermite; and an instantaneous demonstration is given of the properties of the author's formulæ for Sturm's auxiliary functions in determining the real roots of an equation by a method quite irrespective of the theory of the common measure, and depending upon a certain extremely simple but unobserved law of quadratic forms, which he terms the law of *inertia*. In place of these formulæ it is shown that others greatly more general, and possessing the same properties as regards the determination of the real roots, may be substituted; the known formulæ are, however, the most simple that can be employed. The author then proceeds to inquire as to the nature of the indications afforded by the signs of a series of successive simplified residues, taken between any two functions

independent of one another, instead of standing in the relation of primitive and derivative, as in Sturm's theorem; this leads to the theory of interpositions, of whch it is shown that the Sturmian theorem may be treated (not so much as a particular case) as an easy corollary. In this part, the author obtains an entirely new rule for determining, in an infinite variety of ways, a superior and inferior limit to the real roots of any algebraical equation, whether numerical or literal.

The fourth section is also divided into two parts. In the first part, the index of interposition for two functions of the same degree is shown to be determinable by means of the quadratic form, previously termed the Bezoutiant; and as a corollary, it follows that the number of real roots of an equation of the degree m depends in a direct manner on the number of positive roots in another equation of the degree $m-1$, all of whose roots are real, and the coefficients of which are quadratic combinations of the coefficients of the given equation.

In the second part of this section, the Bezoutiant is considered under a purely morphological point of view. It is shown to be a combinantive invariant of the two given functions (each treated as homogeneous functions of two variables), remaining unaltered when any linear combination of the two given functions is substituted for the functions themselves, and also when any linear substitutions are impressed upon the variables of the given functions, provided that certain corresponding substitutions are impressed upon the variables of the Bezoutiant. The family of forms to which the Bezoutiant belongs is ascertained, and a method given for finding the constituent forms of this family (one less in number than the number of odd integers not exceeding in magnitude the degree of either of the given functions which, throughout this section, are supposed to be of equal dimensions in x), of which all other forms of the family will be numerico-linear functions. The numerical coefficients connecting the Bezoutiant with this constituent group, are calculated for the cases corresponding to any index from 1 to 6 inclusive. Finally, the author remarks upon the different directions in which the subject matter of the ideas involved in Sturm's justly celebrated theorem admits of being expanded, and of which the most promising is, in his opinion, that which leads through the theory of interpositions. Several of the theorems in this memoir have been previously published by the author, but they are here given along with a great deal of new matter in a connected form, and with the demonstrations annexed, for the first time.

4. "On the frequent occurrence of Indigo in Human Urine, and on its Chemical and Physiological Relations." By Arthur Hill Hassall, M.D., Physician to the Royal Free Hospital, &c. Communicated by Professor Sharpey, F.R.S. Received June 9, 1853.

The author was led to the investigations laid before the Society in the above communication, by the following circumstances:—

Some three or four years since, when examining urinary deposits